

# Uniqueness of Some Differential Polynomials of Meromorphic Functions

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## Abstract

In this paper, we prove some uniqueness results which improve and generalize several earlier works. Also, we prove a value distribution result concerning  $f^{(k)}$  which provides a partial answer to a question of Fang and Wang [A note on the conjectures of Hayman, Mues and Gol'dberg, *Comp. Methods, Funct. Theory* (2013) **13**, 533–543].

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# 1 Introduction

Throughout, by a meromorphic function we always mean a non-constant meromorphic function in the complex plane  $\mathbb{C}$ . We use the standard notations of Nevanlinna Theory such as  $m(r, f)$ ,  $N(r, f)$ ,  $T(r, f)$ ,  $S(r, f)$  etc. (one may refer to [3]). Let  $f$  and  $g$  be two meromorphic functions and  $a \in \mathbb{C}$ . By  $E(a, f)$ , we denote the set of zeros of  $f - a$  counting multiplicities (CM) and by  $\overline{E}(a, f)$ , the set of zeros of  $f - a$  ignoring multiplicities (IM). Two meromorphic functions  $f$  and  $g$  are said to share the value  $a$  CM if  $E(a, f) = E(a, g)$  and to share the value  $a$  IM if  $\overline{E}(a, f) = \overline{E}(a, g)$ . Further, by  $E_k(a, f)$ , we denote the set of zeros of  $f - a$  with multiplicities atmost  $k$  in which each zero is counted according to its multiplicity. Also, by  $\overline{E}_k(a, f)$ , we denote the set of zeros of  $f - a$  with multiplicity atmost  $k$ , counted once.

We denote by  $\mathcal{A}$ , the class of meromorphic functions  $f$  satisfying

$$\overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) = S(r, f).$$

Clearly, each member of class  $\mathcal{A}$  is a transcendental meromorphic function. Further, by  $\mathcal{M}(D)$  we denote the space of all meromorphic functions on a domain  $D$ . A mapping  $M : \mathcal{M}(\mathbb{C}) \rightarrow \mathcal{M}(\mathbb{C})$  given by

$$M[f] = a \cdot \prod_{j=0}^k (f^{(j)})^{n_j}; \quad \forall f \in \mathcal{M}(\mathbb{C})$$

with  $n_0, n_1, \dots, n_k$  as non-negative integers and  $a \in \mathcal{M}(\mathbb{C}) : a \not\equiv 0$ ; is called a *differential monomial* of degree  $d = \sum_{j=0}^k n_j$  and weight  $w(M) = \sum_{j=1}^k (1 + n_j)$ . We call  $a$  the co-efficient of  $M$ . If  $a = 1$ , then  $M$  is said to be *normalised*. A sum  $P := \sum_{j=1}^p M_j$  of differential monomials  $M_1, M_2, \dots, M_p$  which are linearly independent over  $\mathcal{M}(\mathbb{C})$  is called a *differential polynomial* of degree

$$\deg(P) := \max\{\deg(M_1), \deg(M_2), \dots, \deg(M_p)\}$$

and the weight

$$w(P) := \max\{w(M_1), w(M_2), \dots, w(M_p)\}.$$

If  $\deg(M_1) = \dots = \deg(M_p) = d$ , we call  $P$ , *homogeneous* (of degree  $d$ ). Also for any  $a \in \mathbb{C}$ , we define

$$N_1\left(r, \frac{1}{f-a}\right) = N\left(r, \frac{1}{f-a}\right) - \overline{N}\left(r, \frac{1}{f-a}\right).$$

and

$$N_2\left(r, \frac{1}{f-a}\right) = \overline{N}\left(r, \frac{1}{f-a}\right) + \overline{N}_{(2)}\left(r, \frac{1}{f-a}\right),$$

where  $N_{(k)}(r, 1/(f - a))$  is the counting function of those zeros of  $f - a$  whose multiplicity is atleast  $k$ , and  $\overline{N}_{(k)}(r, 1/(f - a))$  is the one corresponding to ignoring multiplicity. Finally, by  $S(f)$ , we denote the set of small functions of  $f$ ; that is,

$$S(f) := \{a \mid a \text{ is meromorphic and } T(r, a) = S(r, f) \text{ as } r \rightarrow \infty\}.$$

The uniqueness theory of meromorphic functions has perfected the value distribution theory of Nevanlinna and has a vast range of applications in Complex Analysis. Particularly, uniqueness theory of meromorphic functions has been proved to be a handy tool in dealing with the problems on normal families of meromorphic functions. For recent progress concerning normality, one may refer to [7], [9], and [15]. For recent developments in the uniqueness theory of meromorphic functions (sharing, weighted sharing and q-difference sharing of polynomials), one may refer to [5], [10], [12] and [16].

In the present paper, we prove some uniqueness results which improve and generalize the works of Yang and Yi [13], Wang and Gao [8], and Huang and Huang [4]. Also, a partial answer to a question of Fang and Wang [2] concerning value distribution of  $f^{(k)} - a$ , where  $k \in \mathbb{N}$  and  $a (\neq 0, \infty)$  is a small function of  $f$ , is obtained.

## 2 Main Results

Yang and Yi [13, Theorem 3.29, p.197] proved the following result for class  $\mathcal{A}$ :

**Theorem 2.1.** *Let  $f, g \in \mathcal{A}$ , and  $a$  be a non-zero complex number. Furthermore, let  $k$  be a positive integer.*

- (i) *If  $\overline{E}_1(a, f) = \overline{E}_1(a, g)$ , then  $f \equiv g$  or  $f.g \equiv a^2$ .*
- (ii) *If  $\overline{E}_1(a, f^{(k)}) = \overline{E}_1(a, g^{(k)})$ , then  $f \equiv g$  or  $f^{(k)}.g^{(k)} \equiv a^2$ .*

A function  $f$  is said to *share a value  $a$  partially with  $g$  IM* if  $\overline{E}(a, f) \subseteq \overline{E}(a, g)$ . We use the notation  $N_1(r, 1/(g - a)|f \neq a)$ , to denote the simple zeros of  $f - a$ , that are not the zeros of  $g - a$ . Using this notation and the notion of partial sharing, we improve Theorem 2.1 as:

**Theorem 2.2.** *Let  $f, g \in \mathcal{A}$ ,  $a$  be a non-zero complex number and  $k$  be a positive integer.*

- (i) *If  $\overline{E}_1(a, f) \subseteq \overline{E}_1(a, g)$  and  $N_1(r, 1/(g - a)|f \neq a) = S(r, g)$ , then  $f \equiv g$  or  $f.g \equiv a^2$ .*
- (ii) *If  $\overline{E}_1(a, f^{(k)}) \subseteq \overline{E}_1(a, g^{(k)})$  and  $N_1(r, 1/(g^{(k)} - a)|f^{(k)} \neq a) = S(r, g)$ , then  $f \equiv g$  or  $f^{(k)}.g^{(k)} \equiv a^2$ .*

**Example.** Consider  $f(z) = e^z$  and  $g(z) = e^{2z}$ . Then  $f, g \in \mathcal{A}$ ,  $\overline{E}_1(1, f) \subseteq \overline{E}_1(1, g)$  and  $N_1(r, 1/(g - 1)|f \neq 1) \neq S(r, g)$ , and the conclusion of Theorem 2.2 does not hold. Thus, the condition “ $N_1(r, 1/(g - a)|f \neq a) = S(r, g)$ ” in Theorem 2.2, is essential.

Theorem 2.2 also holds if we take  $a$  to be a small function different from 0 and  $\infty$ , as in that case we can take functions  $F = f/a$  and  $G = g/a$  instead of  $f$  and  $g$  so that  $F, G \in \mathcal{A}$ .

In 2011, Huang and Huang [4, Theorem 3, p. 231] improved a result of Yang and Hua [11, Theorem 1, p. 396] as

**Theorem 2.3.** *Let  $f$  and  $g$  be two meromorphic functions and  $n \geq 19$  be an integer. If  $E_1(1, f^n f') = E_1(1, g^n g')$ , then either  $f = dg$  for some  $(n+1)$ th root of unity  $d$  or  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{(n+1)} c^2 = -1$ .*

In this paper, we improve Theorem 2.3 for functions of class  $\mathcal{A}$  as

**Theorem 2.4.** *Let  $f, g \in \mathcal{A}$ ,  $n \geq 2$  be an integer and  $a(\neq 0) \in \mathbb{C}$ . If  $\overline{E}_1(a, f^n f') = \overline{E}_1(a, g^n g')$ , then either  $f = dg$  for some  $(n+1)$ th root of unity  $d$  or  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{(n+1)} c^2 = -a^2$ .*

Concerning sharing of small functions, Wang and Gao [8, Theorem 1.3, p.2] proved

**Theorem 2.5.** *Let  $f$  and  $g$  be two transcendental meromorphic functions,  $a(\neq 0) \in S(f) \cap S(g)$ , and let  $n \geq 11$  be positive integer. If  $f^n f'$  and  $g^n g'$  share a CM, then either  $f^n f' g^n g' \equiv a^2$ , or  $f = tg$  for a constant such that  $t^{n+1} = 1$ .*

Here in this paper, we partially extend this result to a more general class of differential polynomials as

**Theorem 2.6.** *Let  $f$  and  $g$  be two transcendental meromorphic functions,  $a(\neq 0) \in S(f) \cap S(g)$ , and let  $n, m, k$  be positive integers satisfying  $n > km + 3m + 2k + 8$ , and  $m > k - 1$ . If  $f^n (f^m)^{(k)}$  and  $g^n (g^m)^{(k)}$  share a CM, then either*

$$f^n (f^m)^{(k)} g^n (g^m)^{(k)} \equiv a^2 \text{ or } f^n (f^m)^{(k)} \equiv g^n (g^m)^{(k)}.$$

For  $m > k - 1$ , we have  $n > k^2 + 4k + 5$  so that by substituting  $k = 1$ , we get  $n > 10$ . Thus Theorem 2.6 reduces to Theorem 2.5.

For the differential polynomials, Barker and Singh [1, Theorem 3, p.190] proved

**Theorem 2.7.** *The differential equation*

$$a f^n f' + P_{n-1}(f) = 0,$$

*where  $a(\neq 0) \in S(f)$  has no transcendental meromorphic solution  $f$  satisfying  $N(r, f) = S(r, f)$ , where  $P_{n-1}(f)$  is a homogeneous differential polynomial of degree  $n - 1$ .*

In a similar way, we can prove the following more general result

**Theorem 2.8.** *The differential equation*

$$af^n(f^m)^{(k)} + P_{n-1}(f) = 0,$$

where  $a(\neq 0) \in S(f)$  and  $m, n$  are positive integers, has no transcendental meromorphic solution  $f$  satisfying  $N(r, f) = S(r, f)$ , where  $P_{n-1}(f)$  is a homogeneous differential polynomial of degree  $n - 1$ .

Concerning the value distribution of  $k$ th derivative of a meromorphic function, Fang and Wang [2, Proposition 3, p.542] proved the following result:

**Theorem 2.9.** *Let  $f$  be a transcendental meromorphic function having at most finitely many simple zeros. Then  $f^{(k)}$  takes on every non-zero polynomial infinitely often for  $k = 1, 2, 3, \dots$*

In the same paper Fang and Wang [2, Question 2, p.543] posed the following question:

**Question:** *Let  $f$  be a transcendental meromorphic function having at most finitely many simple zeros. Must  $f^{(k)}$  take on every non-zero rational function infinitely often for  $k = 1, 2, 3, \dots$  ?*

Here, we give a partial answer to this question involving small function as

**Theorem 2.10.** *Let  $f$  be a transcendental meromorphic function having at most finitely many simple zeros and  $N(r, 1/f'') = S(r, f)$ . Let  $a(\neq 0, \infty) \in S(f)$ , then  $f^{(k)} - a$  has infinitely many zeros for  $k = 1, 2, 3, \dots$*

### 3 Some Lemmas

We recall the following results which we shall use in the proof of main results of this paper:

**Lemma 3.1.** [11, Theorem 3, p.396] *Let  $f$  and  $g$  be two non-constant entire functions,  $n \geq 1$  and  $a(\neq 0) \in \mathbb{C}$ . If  $f^n f' g^n g' = a^2$ , then  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{(n+1)} c^2 = -a^2$ .*

**Lemma 3.2.** [13, Lemma 1.10, p.82] *Let  $f_1$  and  $f_2$  be non-constant meromorphic functions and  $c_1, c_2$  and  $c_3$  be non-zero constants. If  $c_1 f_1 + c_2 f_2 \equiv c_3$ , then*

$$T(r, f_1) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r, f_1).$$

**Lemma 3.3.** [13, Lemma 3.8, p.193] *If  $f \in \mathcal{A}$  and  $k$  is a positive integer, then  $f^{(k)} \in \mathcal{A}$ .*

**Lemma 3.4.** [13, Lemma 3.9, p.194] *If  $f, g \in \mathcal{A}$  and  $f^{(k)} = g^{(k)}$ , where  $k$  is a positive integer, then  $f \equiv g$ .*

**Lemma 3.5.** [13, Lemma 3.10, p.194] If  $f \in \mathcal{A}$  and  $a$  is a finite non-zero number, then

$$N_1 \left( r, \frac{1}{f-a} \right) = T(r, f) + S(r, f),$$

where  $N_1(r, 1/(f-a))$  denotes the simple zeros of  $f-a$ .

**Lemma 3.6.** [13, Theorem 1.24, p.39] Suppose  $f$  is a nonconstant meromorphic function and  $k$  is a positive integer. Then

$$N \left( r, \frac{1}{f^{(k)}} \right) \leq N \left( r, \frac{1}{f} \right) + k\overline{N}(r, f) + S(r, f).$$

**Lemma 3.7.** [8, Lemma 2.3, p.3] Let  $f$  and  $g$  be two meromorphic functions. If  $f$  and  $g$  share 1 CM, then one of the following must occur:

- (i)  $T(r, f) + T(r, g) \leq 2\{N_2(r, 1/f) + N_2(r, 1/g) + N_2(r, f) + N_2(r, g)\} + S(r, f) + S(r, g)$ ,
- (ii) either  $f \equiv g$  or  $fg \equiv 1$ .

**Lemma 3.8.** [2, Lemma 1, p.537] Let  $f$  be a transcendental meromorphic function,  $k \geq 2$  be an integer, and  $\epsilon > 0$ . Then

$$(k-1)\overline{N}(r, f) + N_1 \left( r, \frac{1}{f} \right) \leq N \left( r, \frac{1}{f^{(k)}} \right) + \epsilon T(r, f).$$

## 4 Proof of Main Results

**Proof.** [Proof of Theorem 2.2] Since  $\overline{E}_1(a, f) \subseteq \overline{E}_1(a, g)$ ,

$$N_1 \left( r, \frac{1}{f-a} \right) \leq N_1 \left( r, \frac{1}{g-a} \right).$$

Since (by Lemma 3.5)

$$N_1 \left( r, \frac{1}{f-a} \right) = T(r, f) + S(r, f)$$

and

$$N_1 \left( r, \frac{1}{g-a} \right) = T(r, g) + S(r, g),$$

therefore,

$$N_2 \left( r, \frac{1}{f-a} \right) = S(r, f),$$

$$N_2 \left( r, \frac{1}{g-a} \right) = S(r, g)$$

and

$$T(r, g) \geq T(r, f) + S(r, f). \quad (4.1)$$

Define a function  $h : \mathbb{C} \rightarrow \overline{\mathbb{C}}$  by

$$h(z) = \frac{f(z) - a}{g(z) - a}. \quad (4.2)$$

Since  $\overline{E}_1(a, f) \subseteq \overline{E}_1(a, g)$ , we have

$$\overline{N}(r, h) \leq \overline{N}(r, f) + \overline{N}_{(2)}\left(r, \frac{1}{g-a}\right) + N_1\left(r, \frac{1}{g-a} | f \neq a\right) = S(r, g) \quad (4.3)$$

$$\overline{N}(r, \frac{1}{h}) \leq \overline{N}(r, g) + \overline{N}_{(2)}\left(r, \frac{1}{f-a}\right) = S(r, g) \quad (4.4)$$

and

$$T(r, h) \leq T(r, f) + T(r, g) + O(1) \leq 2T(r, g) + S(r, g).$$

Let  $f_1 = (1/a)f$ ,  $f_2 = h$ ,  $f_3 = (-1/a)hg$ . Then,

$$\sum_{j=1}^3 f_j \equiv 1. \quad (4.5)$$

Combining (4.2), (4.3) and (4.4), we get

$$\sum_{j=1}^3 \left( \overline{N}(r, f_j) + \overline{N}(r, \frac{1}{f_j}) \right) = S(r, g).$$

Clearly,  $f_1, f_2$  and  $f_3$  are linearly dependent and so there exist three constants  $c_1, c_2$  and  $c_3$  (atleast one of them is not zero) such that

$$\sum_{j=1}^3 c_j f_j = 0 \quad (4.6)$$

If  $c_1 = 0$ , then from (4.6) we see that  $c_2 \neq 0$ ,  $c_3 \neq 0$ , and

$$f_3 = -\frac{c_2}{c_3} f_2. \quad (4.7)$$

Substituting (4.7) into (4.5) gives

$$f_1 + (1 - \frac{c_2}{c_3}) f_2 = 1. \quad (4.8)$$

From (4.7) and (4.8), we get

$$T(r, f_3) = T(r, f_1) + O(1)$$

and thus

$$T(r) = T(r, f_1) + O(1) \quad (4.9)$$

where  $T(r) = \max_{1 \leq j \leq 3} \{T(r, f_j)\}$ .

Since  $f_1$  is not a constant, it follows from (4.8) that  $1 - c_2/c_3 \neq 0$ . From (4.8), (4.9) and Lemma 3.2, we deduce that

$$T(r) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r) = S(r),$$

where  $S(r) = o(T(r))$ , which is a contradiction and so  $c_1 \neq 0$ , and then (4.6) gives

$$f_1 = -\frac{c_2}{c_1}f_2 - \frac{c_3}{c_1}f_3. \quad (4.10)$$

Now, from (4.5) and (4.10), we get

$$\left(1 - \frac{c_2}{c_1}\right)f_2 + \left(1 - \frac{c_3}{c_1}\right)f_3 = 1. \quad (4.11)$$

We consider the following three cases:

*Case 1:*  $1 - c_2/c_1 \neq 0$  and  $1 - c_3/c_1 \neq 0$ .

In this case, (4.10) and (4.11) gives

$$f_1 = \frac{c_2 - c_3}{c_1 - c_2}f_3 - \frac{c_2}{c_1 - c_2}. \quad (4.12)$$

From (4.11) and (4.12), we have

$$T(r, f_2) = T(r, f_1) + O(1)$$

and hence

$$T(r) = T(r, f_1) + O(1). \quad (4.13)$$

Applying Lemma 3.2 to (4.11) and using (4.13), we obtain

$$T(r) < \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_2) + S(r) = S(r),$$

which is a contradiction.

*Case 2:*  $1 - c_2/c_1 = 0$ .

From (4.11), we have  $1 - c_3/c_1 \neq 0$ , and

$$f_3 = \frac{c_1}{c_1 - c_3}. \quad (4.14)$$

Since  $1 - c_2/c_1 = 0$ , we obtain  $c_1 = c_2$ . Thus from (4.10) and (4.14), we obtain

$$f_1 + f_2 = -\frac{c_3}{c_1 - c_3}. \quad (4.15)$$

If  $c_3 \neq 0$ , then by applying Lemma 3.2 to (4.15), we obtain

$$T(r) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_2}\right) + \overline{N}(r, f_1) + S(r) = S(r),$$



which is a contradiction. Hence  $c_3 = 0$  and so from (4.14), it follows that  $f_3 \equiv 1$ .

*Case 3:*  $1 - c_3/c_1 = 0$ .

From (4.11), we have  $1 - c_2/c_1 \neq 0$ , and

$$f_2 = \frac{c_1}{c_1 - c_2}. \quad (4.16)$$

Since  $1 - c_3/c_1 = 0$ , we obtain  $c_1 = c_3$ . Thus from (4.10) and (4.16), we obtain

$$f_1 + f_3 = -\frac{c_2}{c_1 - c_1}. \quad (4.17)$$

If  $c_2 \neq 0$ , then by applying Lemma 3.2 to (4.17), we obtain

$$T(r) < \overline{N}\left(r, \frac{1}{f_1}\right) + \overline{N}\left(r, \frac{1}{f_3}\right) + \overline{N}(r, f_1) + S(r) = S(r),$$

which is a contradiction. Hence  $c_2 = 0$  and so from (4.16), it follows that  $f_2 \equiv 1$ .

Thus if  $f_2 \equiv 1$ , then by (4.2), we get,  $f \equiv g$ . If  $f_3 \equiv 1$ , then (4.2) gives  $f.g \equiv a^2$ . This proves (i).

From Lemma 3.3, we see that  $f^{(k)}, g^{(k)} \in A$ . Using the conclusion of (i), we get, either

$$f^{(k)} \equiv g^{(k)}$$

or

$$f^{(k)}.g^{(k)} \equiv a^2.$$

If  $f^{(k)} \equiv g^{(k)}$ , then from Lemma 3.4, we have  $f \equiv g$ . This completes the proof of (ii).  $\square$

**Proof.** [Proof of Theorem 2.4] Let the functions  $F$  and  $G$  be given by

$$F = \frac{f^{n+1}}{n+1} \text{ and } G = \frac{g^{n+1}}{n+1}.$$

By hypothesis,  $\overline{E}_1(a, f^n f') = \overline{E}_1(a, g^n g')$ , therefore

$$\overline{E}_1(a, F') = \overline{E}_1(a, G').$$

Now

$$\begin{aligned} \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) &= \overline{N}\left(r, \frac{f^{n+1}}{n+1}\right) + \overline{N}\left(r, \frac{n+1}{f^{n+1}}\right) \\ &= \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f}\right) \\ &= S(r, f) \\ &= S(r, F). \end{aligned}$$

Similarly by replacing  $F$  by  $G$  in above equation, we have

$$\overline{N}(r, G) + \overline{N}(r, \frac{1}{G}) = S(r, G).$$

Thus  $F, G \in \mathcal{A}$  and so by the Theorem 2.1, it follows that either

$$F'G' \equiv a^2 \text{ or } F \equiv G.$$

Consider the case  $F'G' \equiv a^2$ , that is,

$$f^n f' g^n g' \equiv a^2. \quad (4.18)$$

Suppose that  $z_1$  is a pole of  $f$  of order  $p$ . Then  $z_1$  is a zero of  $g$  of order say  $q$  and so from (4.27), we find that

$$nq + q - 1 = np + p + 1.$$

That is,  $(q - p)(n + 1) = 2$ , which is not possible as  $n \geq 2$  and  $p, q$  are positive integers. Thus  $f$  and  $g$  are entire functions and so from Lemma 3.1, we get  $f(z) = c_1 e^{cz}$  and  $g(z) = c_2 e^{-cz}$ , where  $c, c_1, c_2$  are constants satisfying  $(c_1 c_2)^{(n+1)} c^2 = -a^2$ .

Next consider the case when  $F \equiv G$ . This gives

$$\frac{f^{n+1}}{n+1} = \frac{g^{n+1}}{n+1},$$

or

$$f^{n+1} = g^{n+1}.$$

Hence  $f = dg$  for some  $(n+1)th$  root of unity  $d$ .

□

**Proof.** [Proof of Theorem 2.6] Let the functions  $F$  and  $G$  be given by

$$F = \frac{f^n (f^m)^{(k)}}{a} \text{ and } G = \frac{g^n (g^m)^{(k)}}{a}.$$

Since  $f^n (f^m)^{(k)}$  and  $g^n (g^m)^{(k)}$  share  $a$  CM,  $F$  and  $G$  share 1 CM. Since (by Lemma 3.6 and  $T(r, a) = S(r, f)$ ),

$$\begin{aligned}
N_2\left(r, \frac{1}{F}\right) + N_2(r, F) &\leq N_2\left(r, \frac{1}{f^n(f^m)^{(k)}}\right) + N_2\left(r, f^n(f^m)^{(k)}\right) + S(r, f) \\
&\leq N_2\left(r, \frac{1}{f^n}\right) + N_2\left(r, \frac{1}{(f^m)^{(k)}}\right) + 2\overline{N}\left(r, f^n(f^m)^{(k)}\right) + S(r, f) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{(f^m)^{(k)}}\right) + 2\overline{N}(r, f) + S(r, f) \\
&\leq 2\overline{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^m}\right) + k\overline{N}(r, f^m) + 2\overline{N}(r, f) + S(r, f) \\
&= 2\overline{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + 2\overline{N}(r, f) + S(r, f) \\
&= 2\overline{N}\left(r, \frac{1}{f}\right) + mN\left(r, \frac{1}{f}\right) + (k+2)\overline{N}(r, f) + S(r, f) \\
&\leq 2T(r, f) + mT(r, f) + (k+2)T(r, f) + S(r, f) \\
&= (k+m+4)T(r, f) + S(r, f),
\end{aligned}$$

therefore,

$$N_2\left(r, \frac{1}{F}\right) + N_2(r, F) \leq (k+m+4)T(r, f) + S(r, f). \quad (4.19)$$

On the similar lines we can write (4.19) for the function  $G$  as

$$N_2\left(r, \frac{1}{G}\right) + N_2(r, G) \leq (k+m+4)T(r, g) + S(r, g). \quad (4.20)$$

Since

$$\begin{aligned}
nT(r, f) = T(r, f^n) &= T\left(r, \frac{f^n(f^m)^{(k)}}{a} \cdot \frac{a}{(f^m)^{(k)}}\right) \\
&\leq T(r, F) + T\left(r, \frac{1}{(f^m)^{(k)}}\right) + T(r, a) + S(r, f) \\
&\leq T(r, F) + T\left(r, \frac{1}{(f^m)^{(k)}}\right) + S(r, f) \\
&\leq T(r, F) + (k+1)T\left(r, \frac{1}{f^m}\right) + S(r, f) \\
&= T(r, F) + (km+m)T\left(r, \frac{1}{f}\right) + S(r, f),
\end{aligned}$$

therefore

$$(n - km - m)T(r, f) \leq T(r, F) + S(r, f). \quad (4.21)$$

Similarly,

$$(n - km - m)T(r, g) \leq T(r, G) + S(r, g). \quad (4.22)$$

Adding (4.21) and (4.22), we get

$$(n - km - m)\{T(r, f) + T(r, g)\} \leq \{T(r, F) + T(r, G)\} + S(r, f) + S(r, g). \quad (4.23)$$

Suppose that

$$T(r, F) + T(r, G) \leq 2\{N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r, F) + N_2(r, G) + S(r, F) + S(r, G)\}. \quad (4.24)$$

holds. Then from (4.19), (4.20), (4.23) and (4.24), we have

$$\begin{aligned} (n - km - m)\{T(r, f) + T(r, g)\} &\leq 2\{N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + N_2(r, F) + N_2(r, G)\} \\ &\quad + S(r, f) + S(r, g). \\ &\leq 2(k + m + 4)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g). \\ &= (2k + 2m + 8)\{T(r, f) + T(r, g)\} + S(r, f) + S(r, g), \end{aligned}$$

which implies that

$$(n - km - 3m - 2k - 8)\{T(r, f) + T(r, g)\} \leq S(r, f) + S(r, g),$$

a contradiction since  $n > km + 3m + 2k + 8$ , where  $m > k - 1$ .

Thus, by Lemma 3.7, it follows that either

$$F.G \equiv 1$$

or

$$F \equiv G.$$

That is, either

$$f^n(f^m)^{(k)}g^n(g^m)^{(k)} \equiv a^2$$

or

$$f^n(f^m)^{(k)} = g^n(g^m)^{(k)}.$$

□

**Proof.** [Proof of Theorem 2.10] Since

$$\begin{aligned} m\left(r, \frac{1}{f}\right) &= m\left(r, \frac{f^{(k)}}{f} \cdot \frac{1}{f^{(k)}}\right) \\ &\leq m\left(r, \frac{1}{f^{(k)}}\right) + m\left(r, \frac{f^{(k)}}{f}\right) \\ &= m\left(r, \frac{1}{f^{(k)}}\right) + S(r, f), \end{aligned}$$

therefore,

$$T(r, f) - N\left(r, \frac{1}{f}\right) \leq T(r, f^{(k)}) - N\left(r, \frac{1}{f^{(k)}}\right) + S(r, f),$$

and so

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f). \quad (4.25)$$

Applying second fundamental theorem of Nevanlinna [3, Theorem 2.5, p.47] to the function  $f^{(k)}$ , we get

$$T(r, f^{(k)}) \leq \overline{N}(r, f^{(k)}) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f^{(k)}).$$

That is,

$$T(r, f^{(k)}) \leq \overline{N}(r, f) + \overline{N}\left(r, \frac{1}{f^{(k)}}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f). \quad (4.26)$$

Since  $N(r, 1/f'') = S(r, f)$ , it follows from Lemma 3.8 with  $k = 1$  that

$$\begin{aligned} \overline{N}(r, f) + N_1\left(r, \frac{1}{f}\right) &\leq N\left(r, \frac{1}{f''}\right) + \epsilon T(r, f) + S(r, f) \\ &= \epsilon T(r, f) + S(r, f). \end{aligned}$$

Thus, from (4.25), (4.26) and the fact that  $f$  has finitely many simple zeros, we get

$$\begin{aligned} T(r, f) &\leq \overline{N}\left(r, \frac{1}{f^{(k)} - a}\right) + \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f^{(k)} - a}\right) + \overline{N}(r, f) + N\left(r, \frac{1}{f}\right) + S(r, f) \\ &= N\left(r, \frac{1}{f^{(k)} - a}\right) + \overline{N}(r, f) + N_1\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f^{(k)} - a}\right) + \epsilon T(r, f) + \frac{1}{2}N\left(r, \frac{1}{f}\right) + S(r, f) \\ &\leq N\left(r, \frac{1}{f^{(k)} - a}\right) + \epsilon T(r, f) + \frac{1}{2}T(r, f) + S(r, f) \\ &= N\left(r, \frac{1}{f^{(k)} - a}\right) + \left(\frac{1}{2} + \epsilon\right)T(r, f) + S(r, f), \end{aligned}$$

which implies that

$$\left(\frac{1}{2} - \epsilon\right)T(r, f) \leq N\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f). \quad (4.27)$$

Taking  $\epsilon = 1/4$  in (4.27), we get

$$T(r, f) \leq 4N\left(r, \frac{1}{f^{(k)} - a}\right) + S(r, f).$$

Hence  $f^{(k)} - a$  has infinitely many zeros for  $k = 1, 2, 3, \dots$

□

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